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Final Exam : MTH 221, Spring 2016

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100/100

- Make sure you have 9 different pages.
- Throughout the exam, write your solution clearly. Otherwise points will be deducted.
- Mobiles are not allowed in this exam.

$\boxed{Q1}$ (10 pts) Consider the system of equations

$$\begin{cases} -3x + 4y &= 8 \\ 6x + ay &= b \end{cases}$$

where a and b represent some real numbers. Find the values for a and b so that the system has

i) a uniques solution,

ii) no solution,

iii) infinitely many solutions.

$$\left[\begin{array}{cc|c} -3 & 4 & 8 \\ 6 & a & b \end{array}\right] \overset{2R_1 + R_2 \to R_2}{\sim} \left[\begin{array}{cc|c} -3 & 4 & 8 \\ 0 & 8+a & 16+b \end{array}\right]$$

i) unique solution determinant $\neq 0$

$-3(8+a) \neq 0$

$\therefore a \neq -8 \qquad b \in \mathbb{R}$

ii) no solution

$a = -8$

$\cancel{b \in \mathbb{R}}$ $b \neq -16$

iii) inifintely many solutions

$a = -8$

$\cancel{b \neq 16}$ $b = -16$

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Q2 **(12 pts)** Let

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 3 \end{bmatrix}.$$

i) **(6 pts)** Find the inverse of the matrix A.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \overset{R_1+R_2\to R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{R_2+R_3\to R_3}{\overset{R_2+R_1\to R_1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \underset{4R_3+R_1\to R_1}{\overset{2R_3+R_2\to R_2}{\sim}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & 5 & 4 \\ 0 & 1 & 0 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\therefore\ A^{-1} = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

ii) **(3 pts)** Find a 3×3 matrix C such that $CA = 2I_3$, where I_3 is the 3×3 identity matrix.

$$C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 6 & 5 & 4 \\ 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 10 & 8 \\ 6 & 6 & 4 \\ 2 & 2 & 2 \end{bmatrix}$$

iii) **(3 points)** Let $Q = (0, 0, 2)$ Find the solution set to $AX = Q^T$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cancel{\underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}\underset{3\times 3}{\begin{bmatrix} 6 & 4 & 4 \\ 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}}} = \underset{3\times 3}{\begin{bmatrix} 6 & 5 & 4 \\ 3 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}}\underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 8 \\ 4 \\ 2 \end{bmatrix}$$

Q3 (10 pts) Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

i) (4 pts) Find all eigenvalues of A.

$$\det(\alpha I_3 - A) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} - \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha+1 & 0 & -1 \\ -3 & \alpha & +3 \\ -1 & 0 & \alpha+1 \end{bmatrix}$$

$$= \alpha(-1)^4 \begin{vmatrix} \alpha+1 & -1 \\ -1 & \alpha+1 \end{vmatrix} = \alpha[(\alpha+1)(\alpha+1) - 1]$$

$$= (\alpha^2+\alpha)(\alpha+1) - 1$$

$$= \alpha^3+\alpha^2+\alpha^2+\alpha-\alpha = 0$$

$$= \alpha^3+2\alpha^2$$

$$= \alpha^2(\alpha+2) = 0$$

$$\boxed{\alpha = 0, \alpha = -2}$$

ii) (5 pts) For each eigenvalue α find a basis for the eigenspace E_α.

$$E_0 = N\begin{bmatrix} 1 & 0 & -1 \\ -3 & 0 & 3 \\ -1 & 0 & 1 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -3 & 0 & 3 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} 3R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_2 \in \mathbb{R}$, $x_3 \in \mathbb{R}$

$$x_1 - x_3 = 0$$
$$x_1 = x_3$$

$$E_0 = \{(x_3, x_2, x_3) \mid x_3, x_2 \in \mathbb{R}\}$$

$$= \text{Span}\{(1,0,1), (0,1,0)\}$$

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$$E_{-2} = N\begin{bmatrix} -1 & 0 & -1 \\ -3 & -2 & 3 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ -3 & -2 & 3 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & -2 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\cancel{x_2 \in \mathbb{R}}$

$\cancel{x_3 = 0}$

$\cancel{-x_1 - x_3 = 0}$

$\cancel{\to x_1 = 0}$

$\cancel{E_{-2} = \{(0, x_2, 0) \mid x_2 \in \mathbb{R}\}}$

$\cancel{= \text{span}\{(0,1,0)\}}$

$x_3 \in \mathbb{R}$

$-2x_2 + 6x_3 = 0$

$\to x_2 = \dfrac{-6x_3}{-2}$

$= 3x_3$

$-x_1 - x_3 = 0$

$x_1 = -x_3$

$$E_{-2} = \{(-x_3, 3x_3, x_3) \mid x_3 \in \mathbb{R}\}$$

$$= \text{span}\{(-1, 3, 1)\}$$

iii) **(1 pts)** Is A diagonalizable? If so, find a matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Yes, multipicaty same as dim

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

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$\boxed{Q4}$ **(10 pts)** Let $S = span\{(1,1,1,0),(1,2,1,0),(-1,1,0,0)\}$ be a subspace of R^4 such that $dim(S) = 3$. Apply Gram-Schmidt process to find an orthogonal basis for S.

$$\text{Basis} = \{\underbrace{(1,1,1,0)}_{V_1}, \underbrace{(1,2,1,0)}_{V_2}, \underbrace{(-1,1,0,0)}_{V_3}\}$$

$$W_1 = V_1 = (1,1,1,0)$$

$$W_2 = V_2 - \left[\frac{V_2 \cdot W_1}{\|W_1\|^2}\right] W_1$$

$$W_3 = V_3 - \left[\frac{V_3 \cdot W_1}{\|W_1\|^2}\right] W_1 - \left[\frac{V_3 \cdot W_2}{\|W_2\|^2}\right] W_2$$

$$W_2 = (1,2,1,0) - \left[\frac{1+2+1}{3}\right](1,1,1,0)$$

$$= (1,2,1,0) - \left(\tfrac{4}{3}, \tfrac{4}{3}, \tfrac{4}{3}, 0\right)$$

$$= \left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3}, 0\right)$$

$$W_3 = (-1,1,0,0) - \left[\overbrace{-1+1}^{0}\right] W_1 - \left[\frac{3}{2}\right]\left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3}, 0\right)$$

$$= (-1,1,0,0) - \left(-\tfrac{1}{2}, 1, -\tfrac{1}{2}, 0\right)$$

$$= \left(-\tfrac{1}{2}, 0, \tfrac{1}{2}, 0\right)$$

$$\text{Orthogonal Basis} = \{(1,1,1,0), \left(-\tfrac{1}{3}, \tfrac{2}{3}, -\tfrac{1}{3}, 0\right), \left(-\tfrac{1}{2}, 0, \tfrac{1}{2}, \ldots\right.$$

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$\boxed{Q5}$ **(10 points)**

(i) **(8 pts)** Let $W = span\{(1,2,3,2),(0,2,3,-7),(2,6,9,-3),(1,8,12,-19)\}$. Find a basis for W. What is $dim(W)$?

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 3 & -7 \\ 2 & 6 & 9 & -3 \\ 1 & 8 & 12 & -19 \end{bmatrix} \xrightarrow[-R_1+R_4 \to R_4]{-2R_1+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 3 & -7 \\ 0 & 2 & 3 & -7 \\ 0 & 6 & 9 & -21 \end{bmatrix}$$

$$\xrightarrow[-3R+R_4 \to R_4]{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 2 & 3 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Basis} = \{(1,2,3,2),(0,2,3,-7)\}$$

or $\{(1,2,3,2),(0,2,3,-7)\}$

$$dim(W) = 2$$

(ii) **(2 pts)** Find a polynomial in P_3, say P, so that $\{x^2+x, x^2+1, P\}$ is a basis for P_3.

$$P = a_1(x^2+x) + a_2(x^2+1) + a_3 P$$

$$P - a_3 P = a_1(x^2+x) + a_2(x^2+1)$$

$$P(1-a_3) = a_1(x^2+x) + a_2(x^2+1)$$

$$P = \frac{a_1(x^2+x) + a_2(x^2+1)}{1-a_3}, \quad a_3 \neq 1$$

for p to be independent it should not be written as a linear combination $\therefore$ $\boxed{P \text{ has to be } 0}$

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Q6 **(13 pts)** Let $T : P_3 \to R^3$ be a linear transformation such that

$$T(a_2x^2 + a_1x + a_0) = (a_2 + a_1 + a_0,\ a_2 + a_1,\ a_2 + a_1 + a_0)$$

.

(i) **(3points)** Find the fake-standard matrix representation.

(ii) **(3points)** Write fake-Ker(T) as a span of a basis.

(iii) **(2points)** Write Ker(T) as a span of a basis.

(iv) **(3points)** Write Range(T) as a span of a basis

(v) **(2points)** Is T one-to-one and onto (isomorphism)? explain.

fake $T : \mathbb{R}^2 \to \mathbb{R}^3$

$$T(a_2, a_1, a_0) = (a_2 + a_1 + a_0,\ a_2 + a_1,\ a_2 + a_1 + a_0)$$

i) fake $M = \begin{array}{c} \begin{array}{ccc} (1,0,0) & (0,1,0) & (0,0,1) \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$

fake ker(T) = Nul ~~fake~~ M $= N\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} -R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{array}$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$a_1 \in \mathbb{R}$

$a_0 = 0$

$a_2 + a_1 + a_0 = 0$

$\Rightarrow a_2 = -a_1$

ii) fake Ker(T) $= \{(-a_1, a_1, 0) \mid a_1 \in \mathbb{R}\}$

$= \text{span}\{(-1, 1, 0)\}$

iii) Ker(T) $=$ ~~span{...}~~

$= \text{span}\{-x^2 + x\}$

iv) Range(T) $=$ Col(M)

$= \text{span}\{(1,1,1), (1,0,1)\}$

v) not 1-1 because dim(Domain) $\neq$ dim(Range)

not onto dim(Range) $\neq$ dim($\mathbb{R}^3$)

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$\boxed{Q7}$ **(2 pts)** Let A be an arbitrary $n \times n$ matrix. Convince me (in at most two lines) that A and A^t have the same eigenvalues.

$|A| = |A^T| \therefore C_\alpha(A) = C_\alpha(A^T)$

$\therefore$ same eigenvalues

$\boxed{Q8}$ **(12 points)**

(i) Explain clearly why $T : R^2 \to R^2$ such that $T(a, b) = (a^2, b + a)$ is not a linear transformation.

$D = \{ (a^2, b+a) | a, b \}$ cannot be written as span

$\text{span} = \{ (a, b), (0, a) \}$ X

(ii) Convince me that $S = \{A \in R^{2\times 2} | rank(A) \le 1\}$ is not a subspace of $R^{2\times 2}$ by showing that one of the subspace axioms fails.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad rank(A) \le 1 \therefore$ at most one leader

$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad A+B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$Rank(A) = 2$

$\therefore$ not a subspace

(iii) Given $S = \{A \in R^{2\times 2} | A^T = A\}$ is a subspace of $R^{2\times 2}$. Write S as a span of a basis.

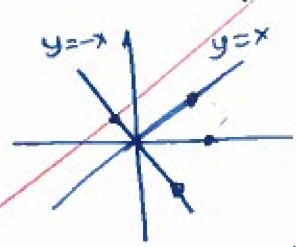
$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = A^T$

$\text{Span} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

(iv) Let S be the set of all points on the line $y = x$ and all points on the line $y = -x$. If S a subspace of R^2, then find a basis. If not, then state clearly why not.

$a = (2, 2)$

$b = (2, -2)$



$(2,2) \quad , (2,-2)$

$a + b = (4, 0) \therefore$ not a subspace because it fails one of the axioms (closure under addition)

$\boxed{Q9}$ **(21 points)**

$A(I_3+2I_3) = |A(3I_3)| =$

(i) If A is a 3×3 matrix with $\det(A)=1$, then $\det(A+2A) =$ ___3___

(ii) If the point (h, 3, 2) is orthogonal to the point (h, 0, -8), then the values of $h =$ ___± 4___

(iii) What are the values of a, b so that $\{x^2+2x+3, -x^2+ax, -x^2-2x+b\}$ is a basis for P_3.

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & a & 0 \\ -1 & -2 & b \end{bmatrix} \underset{R_1+R_3\to R_3}{\overset{R_1+R_2\to R_1}{\sim}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & a+2 & 3 \\ 0 & 0 & b+3 \end{bmatrix}$$

$a \neq -2$

$b \neq -3$

(iv) Let $T: R^2 \to R$ be a linear transformation such that T(1, 1) = 4, T(0, 3) = -2. Find T(5, 17).

$T(5,17) = 5T(1,1) + 4T(0,3)$

$= 5\times4 + 4\times-2 = \underline{12}$

(v) Given A is 3×3 such that $A \xrightarrow{2R_3} B \xrightarrow{2R_1+R_3\to R_3} U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$.

a. Find two elementary matrices say F, W such that $FWA = U$.

$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{2R_3}{\sim} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{2R_1+R_3\to R_3}{\sim}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

b. Find an invertible lower triangular matrix L such that $A = LU$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{-2R_1+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \overset{\frac{1}{2}R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{2} \end{bmatrix}$$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{2} \end{bmatrix}$

(vi) Let A be a 3×3 matrix and $Q=(1,1,0)$, $W=(0,0,5)$. Given $AQ^T = 3Q^T$ and $AW^T = 3W^T$, and $|A| = 36$..
Find $Trace(A^2+2I_3)$.

eigen value of $A \Rightarrow \quad 3\times3\times\alpha = 36$

$\alpha = 4$

α for $(A^2+2I_3) \Rightarrow \quad 3^2+2, \quad 4^2+2$

$\Downarrow \quad \Downarrow$

$11 \quad 18$

$Trace(A^2+2I_3) = 11+11+18 = \underline{40}$

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